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THE DYNAMICAL ANALYSIS OF THE SUSTAINABILITY OF A RECYCLING MATHEMATICAL MODEL

Abstract. *The paper presents a nonlinear system of equations with discrete time and time delays that represents a recycling model. Customers and three different economic agents: the manufacturer, the offline recycler, and the online recycler, made up the closed-loop supply chain. We consider that the economic agents make decisions based on historical data within the context of a repeated game that took into account the marginal profit. The recycling prices of refurbished goods, collected by each agent to the consumer, were taken as variables in our mathematical model. In order to investigate the system's sustainability, a complex dynamic analysis was carried out. To illustrate the theoretical findings, numerical simulations are provided. Finally, conclusions and future directions are drawn.*

Keywords*: Dynamical systems, Equilibrium, Numerical simulations, Recycling, Sustainability*

JEL Classification: C61, C62, Q53, Q56

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1. Introduction

The rapid growth of the economy has increased environmental pollution and thus it is necessary to take measures to reduce its effects. In this context, one of the measures, taken into account by the economic agents, is the option of recycling, by introducing refurbished goods into consumption. The product recycling process has a positive effect on both producers and consumers. More precisely, product recycling helps producers in their sustainable development, while for consumers, the positive effect is related to saving money (Zhu et al., 2022). Recently, the issue of sustainability has become of interest to many researchers (Yamaguchi, 2013; Keirstead, 2014; Dinga, 2016; Dorsey and Hardy, 2018; Vințe et al., 2021; Katariya and Shukla, 2022).

In the framework of dynamic Stackelberg game, in (Ma et al., 2022), the authors study a multichannel supply chain made up of a producer, an online retailer, and a traditional retailer operating under cap-and-trade regulation. Comparisons are made between the supply chains' profits, carbon emissions, and societal welfare in the three distinct scenarios. According to their findings, it is best to keep changes to order amount, consumer channel preferences, and low-carbon preference within a reasonable range. In the same framework of Stackelberg game, in (Qi et al., 2022), the government's optimal subsidy decision and the manufacturer, retailer, and online platform's optimal price decision, are investigated. It is proposed that the online recycling platform should receive direct government funding. In Wang et al. (2022), it is studied how tariff policy affects privatizing state-owned businesses, paying for environmental taxes, reducing pollution, and promoting social welfare in an open economic system. When tariffs rise, the percentage of state-owned company privatization grows, environmental taxes decrease, and environmental degradation decreases.

A closed-loop supply chain based on the development of an online recovery platform is dynamically analysed in (Zhu et al., 2021), which consists of a manufacturer and a third-party recycler, by using chaos theory, complex dynamics theory, and numerical simulation. The recycling market is most significantly impacted by how quickly recycling prices adjust.

The motivation of the paper is based on the fact that sustainability is very important for the economy, biology, and also for the environment. Furthermore, sustainable consumption leads to new requirements for efficient management of the "green" supply chain, which generate great economic and environmental benefits. Thus, we consider a closed-loop supply chain that consists of the manufacturer, the offline recycler, and the online recycler that is illustrated by a mathematical model described by a nonlinear system with discrete time and delay. Four variables were considered: the recycling prices of refurbished goods, collected by each agent to the consumer, and the recycling price of used products collected by agent 3 to the third-party. From the dynamical systems point of view, sustainability occurs when, in the neighborhood of an equilibrium point, the trajectories are oscillating. Therefore, the aim of the paper is to conduct a complex analysis related to the sustainability of the economic system.

The paper proceeds as follows: the discrete mathematical model with time delay is presented in Section 2. The analysis of the corresponding characteristic equation is provided in Section 3. Section 4 presents the Möbius transformation, and the normal form of the extended discrete mathematical model is given in Section 5. The numerical simulations substantiate the theoretical results in Section 6 and finally we have some conclusions and recommendations for further research.

2. Discrete mathematical model with delay with multiple recycling channels

The producer, a third-party online recycler, a third-party offline recycler, and customers make up the closed-loop supply chain in this study (Zhu et al., 2021). Customers have three options to return used goods: through the online or offline recycler third party, or the producer. In addition, the outside online or offline recyclers sell second-hand goods from customers to the producer. After completing the procedure for collecting goods returned by consumers, the manufacturing company reconditions them and resells them to consumers as refurbished items.

In order to write the mathematical model, we consider agent 1 as the online recycle, agent 2 as offline recycle and agent 3 as the producer. The following notations are used:

a:recycling products market size;

b_i: consumers'sensitivity coefficient to the price of agenti'recycling channel, $i = 1,2,3$;

D_i: recycling quantity of the agent, $i = 1,2,3$;

c: remanufacturing cost of refurbished goods;

θ_i: consumer dependence on agent i' recycling channel, i=1,2;

r_{ij}: price cross influence coefficient between agent $i$and agent j, i$\neq j$, i=1,3 and j=1,2;

q_i:recycling price of refurbished goods collected by agent i, $i = 1,2,3$.

The recycling quantity of the three economic agents are given by:

$$\begin{aligned}
D_1 &= \theta_1 a + b_1 q_1 - r_{12}(q_2 - q_1) - r_{31}(q_3 - q_1),\\
D_2 &= \theta_2 a + b_2 q_2 - r_{12}(q_1 - q_2) - r_{32}(q_3 - q_2),\\
D_3 &= (1-\theta_1-\theta_2)a + b_3 q_3 - r_{31}(q_1 - q_3) - r_{32}(q_2 - q_3).
\end{aligned} \tag{1}$$

Suppose $0 \le \theta_1 + \theta_2 \le 1$, $b_1 = b_2 = b_3 = b$ and $r_{12} = r_{31} = r_{32} = r$. We have:

$$\begin{aligned} D_1 &= \theta_1 a + bq_1 - r(q_2 - q_1) - r(q_3 - q_1), \\ D_2 &= \theta_2 a + bq_2 - r(q_1 - q_2) - r(q_3 - q_2), \\ D_3 &= (1 - \theta_1 - \theta_2)a + bq_3 - r(q_1 - q_3) - r(q_2 - q_3). \end{aligned} \tag{2}$$

The profits of the agents, denoted by P_1, P_2 and P_3, are:

$$\begin{aligned} &P_1 = (q_4 - q_1)D_1 - S_1, \qquad P_2 = (q_4 - q_2)D_2 - S_2, \\ &P_3 = (q_5 - q_3 - c)D_3 + (q_5 - q_4 - c)(D_1 + D_2) - S_3 \end{aligned} \tag{3}$$

where q_5 is the retailer price of remanufactured products, q_4 is the recycling price of used products collected by agent 3 to the third-party, and S_i stands for other operating costs of the agent i, i=1,2,3.

Profit in the supply chain can be shown as follows, when:
$P_{31} + P_{32} = P_3$ and $S_{31} + S_{32} = S_3$

$$\begin{aligned} P_1 &= (q_4 - q_1)(\theta_1 a + bq_1 - r(q_2 - q_1) - r(q_3 - q_1)) - S_1, \\ P_2 &= (q_4 - q_2)(\theta_2 a + bq_2 - r(q_1 - q_2) - r(q_3 - q_2)) - S_2, \\ P_{31} &= (q_5 - q_3 - c)((1 - \theta_1 - \theta_2)a + bq_3 - r(q_1 - q_3) - r(q_2 - q_3)) - S_{31} \\ P_{32} &= (q_5 - q_3 - c)((\theta_1 + \theta_2)a + (b + r)(q_1 + q_3) - 2rq_3) - S_{32}. \end{aligned}$$

We adopt the following assumptions to streamline the analysis (Zhu, 2021):

(i) The remanufactured product can be put back on the market for sale after its quality level satisfies the required standard.

(ii) The market is large enough to satisfy the consumption of new and refurbished goods.

(iii) The market supply of refurbished products is based on the direct relationship between quantity and price. Other factors are not taken into account.

(iv) Both producers and independent recyclers are independent rational decision-makers, whose primary concern is maximizing their profit.

In each period, $j = 1, \ldots, n$, the producer dynamically adjusts the price. Based on the marginal profit of the decision in the previous periods, the producer makes the current decision based on the forecast information, if the marginal profit of the producer is positive in period $j-1$, and the strategy is to be continued in the period j. On the other hand, if the producer takes into account the negative marginal profit in period $j-1$, then P_{31}andP_{32} should be adjusted in the period j.

$$
\begin{aligned}
P_1[j] &= (q_4[j]-q_1[j])(\theta_1 a + bq_1[j] - r(q_2[j]-q_1[j]) - r(q_3[j]-q_1[j])) \\
&\quad - S_1, \\
P_2[j] &= (q_4[j]-q_2[j])(\theta_2 a + bq_2[j] - r(q_1[j]-q_2[j]) - r(q_3[j]-q_2[j])) \\
&\quad - S_2, \\
P_{31}[j] &= (q_5[j]-q_3[j]-c)((1-\theta_1-\theta_2)a + bq_3[j] - r(q_1[j]-q_3[j]) \\
&\quad - r(q_2[j]-q_3[j])) - S_{31}, \\
P_{32}[j] &= (q_5[j]-q_3[j]-c)((\theta_1+\theta_2)a + (b+r)(q_1[j]+q_3[j]) - 2rq_3[j]) \\
&\quad - S_{32}.
\end{aligned}
\tag{4}
$$

In the framework of a repeated game, taking into consideration the decision's marginal profit,the economic agentsmake the decisions based on a previous time period. If in the time period j-1, the marginal profit is positive, in period j the strategy will continue. On the other hand, the negative marginal profit leads to adjustingthe process in period j.In what follows we use the variables $q_1, q_2, q_3, q_4$and the corresponding discrete mathematical model is given by:

$$
\begin{aligned}
q_1[j] &= q_1[j-1] + \beta_1 q_1[j-1]\frac{\partial \mathrm{P}_1[j-1]}{\partial \mathrm{q}_1[j-1]}, \\
q_2[j] &= q_2[j-1] + \beta_2 q_2[j-1]\frac{\partial \mathrm{P}_2[j-1]}{\partial \mathrm{q}_2[j-1]}, \\
q_3[j] &= q_3[j-1] + \beta_3 q_3[j-1]\frac{\partial \mathrm{P}_{31}[j-1]}{\partial \mathrm{q}_3[j-1]}, \\
q_4[j] &= q_4[j-1] + \beta_4 q_4[j-1]\frac{\partial \mathrm{P}_{32}[j-1]}{\partial \mathrm{q}_4[j-1]},
\end{aligned}
\tag{5}
$$

where $\beta_i, i = 1,\ldots,4$ are the adjustment parameters.

Then, using (4) and (5) we have:

$$
\begin{aligned}
q_1[j] &= q_1[j-1] + \beta_1 q_1[j-1](-\theta_1 a - bq_1[j-1] + r(q_2[j-1]-q_1[j-1]) + \\
&\quad + r(q_3[j-1]-q_1[j-1]) + (q_4[j-1]-q_1[j-1])(b+2r)), \\
q_2[j] &= q_2[j-1] + \beta_2 q_2[j-1](-\theta_1 a - bq_2[j-1] + r(q_1[j-1]-q_2[j-1]) + \\
&\quad + r(q_3[j-1]-q_1[j-1]) + (q_4[j-1]-q_2[j-1])(b+2r)), \\
q_3[j] &= q_3[j-1] + \beta_3 q_3[j-1](rq_1[j-1] + rq_2[j-1] - 2(b+2r)q_3[j-1] + \\
&\quad + 2rq_4[j-1] - (1-\theta_1-\theta_2)a + (q_5-c)b), \\
q_4[j] &= q_4[j-1] + \beta_4 q_4[j-1](-(b+r)(q_1[j-1] + q_2[j-1] + 2rq_3[j-1] - \\
&\quad - (\theta_1+\theta_2)a).
\end{aligned}
\tag{6}
$$

Moreover, we consider that when the decision on the price for period j is taken, the profits from previous periodsshould be taken into account.We use the transformation $q_i[j] -> q_i[j] + q_{i0}, i = 1,\dots,4$, where $q_{i0}, i=1,\dots,4$ are the coordinates of the stationary stateand we introduce the time delays denoted by τ_2 *for* $q_2[j-\tau_2]$ and τ_3 *for* $q_3[j-\tau_3],\ q_4[j-\tau_3]$. The delays are natural numbers such that $1 \le \tau_2, \tau_3 \le 3$.Therefore, the discrete mathematical model with time delay is given by:

$$\begin{aligned}
q_1[j] &= q_1[j-1] + \beta_1(q_1[j-1] + q_{10})(-\theta_1 a - bq_1[j-1] + r(q_2[j-\tau_2] - q_1[j-1]) + \\
&\quad + r(q_3[j-\tau_3] - q_1[j-1]) + (q_4[j-\tau_3] - q_1[j-1])(b+2r)), \\
q_2[j] &= q_2[j-1] + \beta_2(q_2[j-1] + q_{20})(-\theta_1 a - bq_2[j-\tau_2] + r(q_1[j-1] - q_2[j-\tau_2]) + \\
&\quad + r(q_3[j-\tau_3] - q_1[j-1]) + (q_4[j-\tau_3] - q_2[j-1])(b+2r)), \\
q_3[j] &= q_3[j-1] + \beta_3(q_3[j-1] + q_{30})(rq_1[j-1] + rq_2[j-\tau_2] - 2(b+2r)q_3[j-\tau_3] + \\
&\quad + 2rq_4[j-\tau_3] - (1-\theta_1-\theta_2)a + (q_5 - c)b), \\
q_4[j] &= q_4[j-1] + \beta_4(q_4[j-1] + q_{40})(-(b+r)(q_1[j-1] + q_2[j-\tau_2] + 2rq_3[j-\tau_3]
\end{aligned} \tag{7}$$

where the coordinates of the stationary state, $q_{i0}, i=1,\dots,4$, are the solutions of the algebraic system:

$$\begin{aligned}
&(-2b-4r)q_1 + rq_2 + rq_3 + (b+2r)q_4 = \theta_1 a, \\
&rq_1 + rq_2 + (-2b-4r)q_3 + 2rq_4 = (1-\theta_1-\theta_2)a - (p-c)b, \\
&rq_1 + rq_2 + (-2b-4r)q_3 + 2rq_4 = (1-\theta_1-\theta_2)a - (p-c)b, \\
&(-b-r)q_1 + (-b-r)q_2 + 2rq_3 + 2rq_4 = (\theta_1+\theta_2)a.
\end{aligned} \tag{8}$$

Using Cramer method for linear systems we obtain the solution of (8).
For the case$\tau_2 = 2, \tau_3 = 3$ we use the notations:

$$q_1[j] = u_9[j-1];\ q_2[j] = u_{10}[j-1];\ q_3[j] = u_{11}[j-1]; q_4[j] = u_{12}[j-1];.$$

so that system (7) becomes a discrete-time system without time delay:

$$\begin{aligned}
&u_1[j] = u_9[j-1];\ \ u_5[j] = u_1[j-1]; \\
&u_2[j] = u_{10}[j-1];\ u_6[j] = u_1[j-1]; \\
&u_3[j] = u_{11}[j-1];\ u_7[j] = u_{31}[j-1]; \\
&u_4[j] = u_{12}[j-1];\ u_8[j] = u_4[j-1];
\end{aligned} \tag{8}$$

$$
\begin{aligned}
u_9[j] &= u_9[j-1] + \beta_1 (u_9[j-1] + q_{10})(-bu_1[j-1] + r(u_6[j-1] - u_1[j-1]) + \\
&\quad + r(u_{11}[j-1] - u_1[j-1]) + (u_{12}[j-1] - u_1[j-1])(b+2r)); \\
u_{10}[j] &= u_{10}[j-1] + \beta_2 (u_{10}[j-1] + q_{20})(-bu_6[j-1] + r(u_{11}[j-1] - u_1[j-1]) + \\
&\quad + r(u_{11}[j-1] - q_1[j-1]) + (q_{12}[j-1] - u_6[j-1])(b+2r)); \\
u_{11}[j] &= u_{11}[j-1] + \beta_3 (u_{11}[j-1] + q_{30})(rq_1[j-1] + ru_6[j-1] - \\
&\quad - 2(b+2r)u_{11}[j-1] + 2ru_{12}[j-1]); \\
u_{12}[j] &= u_{12}[j-1] + \beta_4 (u_{12}[j-1] + q_{40})(-(b+r)(u_1[j-1] + \\
&\quad + u_6[j-1]) + 2ru_{11}[j-1]).
\end{aligned} \tag{9}
$$

3. The analysis of the characteristic equation

For simplicity, we use the following notations:

$$t_1 = \beta_1 q_{10};\ t_2 = \beta_2 q_{12};\ t_3 = \beta_3 q_{30};\ t_4 = \beta_4 q_{410}.$$

and we consider the matrices with the first derivatives of the functions from the right part of (9)

$$
A_1 = \begin{pmatrix} a_{91} & 0 & 0 & 0 \\ a_{101} & 0 & 0 & 0 \\ a_{111} & 0 & 0 & 0 \\ a_{121} & 0 & 0 & 0 \end{pmatrix};
A_2 = \begin{pmatrix} 0 & a_{96} & 0 & 0 \\ 0 & a_{106} & 0 & 0 \\ 0 & a_{116} & 0 & 0 \\ 0 & a_{126} & 0 & 0 \end{pmatrix};
A_3 = \begin{pmatrix} a_{99} & 0 & a_{911} & a_{912} \\ 0 & a_{1010} & a_{1011} & a_{1012} \\ 0 & 0 & a_{1111} & a_{1112} \\ 0 & 0 & a_{1211} & a_{1212} \end{pmatrix} \tag{10}
$$

where

$a_{91} = -2(b+2r)t_1$

$a_{96} = rt_1$

$a_{99} = 1$

$a_{911} = rt_1$

$a_{912} = (b+2r)t_1$

$a_{101} = -2rt_2$

$a_{106} = -2(b+r)t_2$

$a_{1010} = 1$

$a_{1011} = 2rt_2$

$a_{1012} = (b+2r)t_2$

$a_{111} = rt_3$

$a_{116} = rt_3$

$a_{1111} = 1 - 2(b+2r)t_3$

$a_{1112} = 2rt_3$

$a_{121} = -(b+r)t_4$

$a_{126} = -(b+r)t_4$

$a_{1211} = 2rt_4$

$a_{1212} = 1$

The corresponding characteristic equation of system (9) is:

$$\det(E_4 z^3 - A_1 z^2 - A_2 z - A_3) = 0 \tag{11}$$

whereE_4 is the identity matrix.

An equivalent form of the characteristic equation is:

$$f_{12}(z) = z^{12} - a_{11}z^{11} - a_{10}z^{10} - a_9 z^9 - a_8 z^8 - a_7 z^7 - a_6 z^6 - \\ -a_5 z^5 - a_4 z^4 - a_3 z^3 - a_2 z^2 - a_1 z - a_0 = 0 \tag{12}$$

where the coefficients are given by:

$a_{11} = 2(b+2r)t_1;\ a_{10} = 2(b+r)t_2;\ a_9 = 2(2b^2 + 6br + 5r^2)t_1t_2 + 2(b+2r)t_3 - 4;$

$a_8 = 4b^2t_1t_3 + 16brt_1t_3 + 15r^2t_1t_3 + b^2t_1t_4 + 3brt_1t_4 - 6(b+2r)t_1;$

$a_7 = 4b_2t_2t_3 + 12brt_2t_3 + 6r^2t_2t_3 + b^2t_2t_4 + 2r^2t_2t_4 - 6(b+r)t_2;$

$a_6 = 8b^3t_1t_2t_3 + 4b^3t_1t_2t_4 + 40b^2rt_1t_2t_3 + 17b_2rt_1t_2t_4 + 62br^2t_1t_2t_3 + 23br^2t_1t_2t_4$
$+ 30r^3t_1t_2t_3 + 10r^3t_1t_2t_4 - 8b^2t_1t_2 - 24brt_1t_2 - 20r^2t_1t_2 - 4r^2t_3t_4 - 6bt_3 - 12rt_3 + 6;$

$a_5 = 2b^3t_1t_3t_4 + 10b^2rt_1t_3t_4 + 8br^2t_1t_3t_4 - 10r^3t_1t_3t_4 - 8b^2t_1t_3 - 2b^2t_1t_4$
$- 32brt_1t_3 - 6brt_1t_4 - 30r^2t_1t_3 - 4r^2t_1t_4 + 6bt_1 + 12rt_1;$

$a_4 = 2b^3t_2t_3t_4 + 10b^2rt_2t_3t_4 + 10br^2t_2t_3t_4 - 8b^2t_2t_3 - 2b^2t_2t_4 - 24brt_2t_3$
$- 6brt_2t_4 - 12r^2t_2t_3 - 4r^2t_2t_4 + 6bt_2 + 6rt_2;$

$a_3 = 8b^4t_1t_2t_3t_4 + 50b^3rt_1t_2t_3t_4 + 102b^2r^2t_1t_2t_3t_4 + 70br^3t_1t_2t_3t_4 - 8b^3t_1t_2t_3$
$- 4b^3t_1t_2t_4 - 40b^2rt_1t_2t_3 - 17b^2rt_1t_2t_4 - 62br^2t_1t_2t_3 - 23br^2t_1t_2t_4 - 30r^3t_1t_2t_3 -$
$10r^3t_1t_2t_4 + 4b^2t_1t_2 + 12brt_1t_2 + 10r^2t_1t_2 + 8r^2t_3t_4 + 6bt_3 + 12rt_3 - 4;$

$a_2 = -2b^3t_1t_3t_4 - 10b^2rt_1t_3t_4 - 8br^2t_1t_3t_4 + 10r^3t_1t_3t_4 + 4b^2t_1t_3 + b^2t_1t_4$
$+ 16brt_1t_3 + 3brt_1t_4 + 15r^2t_1t_3 + 2r^2t_1t_4 - 2bt_1 - 4rt_1;$

$a_1 = -2b^3t_2t_3t_4 - 10b^2rt_2t_3t_4 - 10br^2t_2t_3t_4 + 4b^2t_2t_3 + b^2t_2t_4$
$+ 12brt_2t_3 + 3brt_2t_4 + 6r^2t_2t_3 + 2r^2t_2t_4 - 2bt_2 - 2rt_2;$

$a_0 = -4r^2t_3t_4 - 2bt_3 - 4rt_3 + 1.$

We are interested in finding the conditions for equation $f_{12}(z) = 0$ to admit complex conjugate roots with their modulus equal to 1. We use Möbius transformation of the decomposition of the polynomial that describes the characteristic equation into products of polynomials whose roots have their modulus equal to 1.

We call total sustainability if all the roots are complex conjugate with their modulus is equal to 1. We call partial sustainability of 2nd, 4th or 6th order if the number of the complex conjugate roots with their modulus equal to 1 is 2, 4 or 6, respectively.

In what follows, we describe the determination of the complex conjugate roots with their modulus equal to 1 using Möbius transformation.

4. Möbius transformation

The Mobius transformation $M(z) = \dfrac{z-1}{z+1}$ sends the upper half-plane into the open unit disc and the real line onto the unit circle without the point 1.Let $f(z)$ be a polynomial of p degree and $f^*(z)$ the transformation of $f(z)$ by Möbius transformation given by (Conrad, 2016):

$$f^*(z) = (z+1)^p f\left(\frac{z-1}{z+1}\right).$$

A real root of $f(z)$ is transformed by $M(z)$ into a root of $f^*(z)$ on the unit circle. Conversely, every root of $f(z)$ on the unit circle other than 1 has the form $\frac{z-i}{z+i}$ for some real number z that is a root of $f^*(z)$. We write $f^*(z) = m_{121}(z) + m_{122}(z)i$, where $m_{121}(z)$ and $m_{122}(z)$ have real coefficients and a real root of $f^*(z)$ is the same thing as a common real root of $m_{121}(z)$ and $m_{122}(z)$, that is, a root of the polynomial $\gcd(m_{121}(z); m_{122}(z))$. Therefore, counting roots of $f(z)$ on the unit circle is the same as checking if $f(1)$ is not 0 and counting real roots of $\gcd(m_{122}(z); m_{121}(z))$. (Conrad, 2016).

Using the Möbius transformation, $f_{12}(z)$ can be written as:

$$f^*(z) = m_{121}(z) + m_{122}(z)i\ ,$$

where:

$$\begin{aligned}
m_{121}(z) &= (-a_0 - a_1 - a_{10} - a_{11} - a_2 - a_3 - a_4 - a_5 - a_6 - a_7 - a_8 - a_9 + 1)z^{12} + 1 + \\
&+ (66a_0 + 44a_1 + 26a_{10} + 44a_{11} + 26a_2 + 12a_3 + 2a_4 - 4a_5 - 6a_6 - \\
&- 4a_7 + 2a_8 + 12a_9 - 66)z^{10} + (-495a_0 - 165a_1 - 15a_{10} - 165a_{11} - 15a_2 + \\
&+ 27a_3 + 17a_4 - 5a_5 - 15a_6 - 5a_7 + 17a_8 + 27a_9 + 495)z^8 + (924a_0 - \\
&- 84a_{10} - 84a_2 + 28a_4 - 20a_6 + 28a_8 - 924)z^6 + (-495a_0 + 165a_1 - \\
&- 15a_{10} + 165a_{11} - 15a_2 - 27a_3 + 17a_4 + 5a_5 - 15a_6 + 5a_7 + 17a_8 - \\
&- 27a_9 + 495)z^4 + (66a_0 - 44a_1 + 26a_{10} - 44a_{11} + 26a_2 - 12a_3 + 2a_4 + \\
&+ 4a_5 - 6a_6 + 4a_7 + 2a_8 - 12a_9 - 66)z^2 - a_0 + a_1 - a_{10} + a_{11} - a_2 + a_3 - a_4 + \\
&+ a_5 - a_6 + a_7 - a_8 + a_9;
\end{aligned}$$

$$m_{122}(z) = (-12a_0 - 10a_1 + 8a_{10} + 10a_{11} - 8a_2 - 6a_3 - 4a_4 - 2a_5 + 2a_7 + $$
$$+ 4a_8 + 6a_9 - 12)z^{11} + (220a_0 + 110a_1 - 40a_{10} - 110a_{11} + 40a_2 + 2a_3$$
$$- 12a_4 - 10a_5 + 10a_7 + 12a_8 - 2a_9 + 220)z^9 + (-792a_0 - 132a_1 - 48a_{10}$$
$$+ 132a_{11} + 48a_2 + 36a_3 - 8a_4 - 20a_5 + 20a_7 + 8a_8 - 36a_9 - 792)z^7 +$$
$$(792a_0 - 132a_1 + 48a_{10} + 132a_{11} - 48a_2 + 36a_3 + 8a_4 - 20a_5 + 20a_7$$
$$- 8a_8 - 36a_9 + 792)z^5 + (-220a_0 + 110a_1 + 40a_{10} - 110a_{11} - 40a_2 + 2a_3$$
$$+ 12a_4 - 10a_5 + 10a_7 - 12a_8 - 2a_9 - 220)z^3 + (12a_0 - 10a_1 - 8a_{10}$$
$$+ 10a_{11} + 8a_2 - 6a_3 + 4a_4 - 2a_5 + 2a_7 - 4a_8 + 6a_9 + 12)z;$$

For example, using the values:

$$a = 10,\ b = 0.6,\ c = 5,\ r = 0.4,\ \theta_1 = 0.65,\ \theta_2 = 0.2,\ S_1 = 1, S_2 = 0.1, S_3 = 300,$$
$$p = 30,\ \beta_1 = 0.0085,\ \beta_1 = 0.0015,\ \beta_1 = 0.0025,$$

the coefficients of $f_{12}(z)$ are given by:
$a_0 = 0.6907$, $a_1 = -0.0569$, $a_2 = -0.3195$, $a_3 = -3.0469$, $a_4 = 0.1957$, $a_5 = 1.1047$;
$a_6 = 5.0092$, $a_7 = -0.2223$, $a_8 = -1.2541$, $a_9 = -3.6526$; $a_{10} = 0.0823$, $a_{11} = 0.4669$.

Then, the roots of $m_{122}(z) = 0$ are:

$z_1 = 0.00001$, $z_2 = -0.2632$, $z_3 = 0.2632$, $z_4 = -0.5773$, $z_5 = 0.5773$, $z_6 = -0.9864$, $z_7 = 0.9864$

and the roots of $f_{12}(z) = 0$ on the unit circle

$$\mu(k) = \frac{z_k - i}{z_k + i}, k = 1, \ldots 7,$$

are given by:

$\mu_1 = -1$, $\mu_2 = -0.8704 + 0.4923i$, $\mu_3 = -0.8704 - 0.4923i$, $\mu_4 = -0.50005 + 0.8659i$,
$\mu_5 = -0.50005 + 0.86599i$, $\mu_6 = -0.0136 + 0.9999i$, $\mu_7 = 0.0136 - 0.9999i$.

Thus, the dynamical system is partially sustainable.

In order to visualize the orbits of the system (7) in what follows, we look for the normal form of (9). Firstly, we have to find the eigenvectors, for a certain value of an eigenvalue of the Jacobian, using the first derivatives of the right part of (9). Then, the coefficients of the normal form are determined.

5. Normal form of system (9)

The Jacobian of the system (9) is the matrix denoted by N :

$$N = [\mathrm{O}, \mathrm{O}, \mathrm{E}_4, \mathrm{E}_4, \mathrm{O}, \mathrm{O}, \mathrm{A}_3, \mathrm{A}_2, \mathrm{A}_1] \in \mathrm{M}_{12x12}$$

where O is the null matrix, E_4 is the unit matrix of the fourth order and A_1, A_2, A_3 are given by (10).

The eigenvector q is given by $Nq = \mu q$, where μ is a root of the characteristic equation (12).

Let the components of q be $q = (q_1, q_2, q_3)$,
where
$q_1 = (u_{11}, u_{21}, u_{31}, u_{41})$; $q_2 = (u_{52}, u_{62}, u_{72}, u_{82})$; $q_3 = (u_{93}, u_{103}, u_{113}, u_{123})$.

The coefficients of the normal form denoted by $(g_{20}, g_{11}, g_{02}, g_{21})$ are determined using the second derivatives of the functions from the right part of (9) at $(0, 0, 0,0)$ (Neamțu and Opriș, 2008).

The second derivatives of the functions from the right part of (9) at $(0, 0, 0,0)$ are given by:

$a_{991} = -2(1+2\mathrm{r})\beta_1$; $\mathrm{a}_{996} := \beta_1\mathrm{r}$, $\mathrm{a}_{9911} = \beta_1\mathrm{r}$, $\mathrm{a}_{9912} = \beta_1(\mathrm{b}+2\mathrm{r})$, $\mathrm{a}_{10106} = -2(\mathrm{b}+\mathrm{r})\beta_2$,
$\mathrm{a}_{101011} = 2\beta_2\mathrm{r}$, $\mathrm{a}_{10101} = -2\beta_2\mathrm{r}$; $\mathrm{a}_{101012} = \beta_2(\mathrm{b}+2\mathrm{r})$, $\mathrm{a}_{11111} = \beta_3\mathrm{r}$, $\mathrm{a}_{11116} = \beta_3\mathrm{r}$,
$\mathrm{a}_{111111} = -2(\mathrm{b}+2\mathrm{r})$; $\mathrm{a}_{111112} = 2\beta_3\mathrm{r}$, $\mathrm{a}_{12121} = -\beta_4(\mathrm{b}+\mathrm{r})$, $\mathrm{a}_{12126} = -\beta_4(\mathrm{b}+\mathrm{r})$, $\mathrm{a}_{121211} = 2\beta_4\mathrm{r}$.

The Taylor series of the right part of the functions from system (9) is:

$$F(q) = Nq + \frac{B(q,q)}{2}$$

where $B(q,q) = (0, 0, 0, 0, 0, 0, 0, 0, B_9(q,q), B_{10}(q,q), B_{11}(q,q), B_{12}(q,q))$, N is the Jacobian matrix of the functions from the right part of the functions from (9) at (0,0,0,0) and

$$\mathrm{B}_9 = \mathrm{a}_{991}\mathrm{u}_{11}\mathrm{u}_{93} + \mathrm{a}_{9911}\mathrm{u}_{113}\mathrm{u}_{93} + \mathrm{a}_{9912}\mathrm{u}_{123}\mathrm{u}_{93} + \mathrm{a}_{996}\mathrm{u}_{62}\mathrm{u}_{93};$$
$$\mathrm{B}_{10} = ((\mathrm{a}_{10101}\mathrm{u}_{103}\mathrm{u}_{11} + \mathrm{a}_{101011}\mathrm{u}_{103}\mathrm{u}_{113} + \mathrm{a}_{10106}\mathrm{u}_{103}\mathrm{u}_{62}) + \mathrm{a}_{101011}\mathrm{u}_{103}\mathrm{u}_{113}) + \mathrm{a}_{101012}\mathrm{u}_{103}\mathrm{u}_{123};$$
$$\mathrm{B}_{11} = \mathrm{a}_{11111}\mathrm{u}_{113}\mathrm{u}_{11} + \mathrm{a}_{11116}\mathrm{u}_{113}\mathrm{u}_{62} + \mathrm{a}_{111111}\mathrm{u}_{113}\mathrm{u}_{113} + \mathrm{a}_{111112}\mathrm{u}_{113}\mathrm{u}_{123};$$
$$\mathrm{B}_{12} = \mathrm{a}_{12121}\mathrm{u}_{11}\mathrm{u}_{123} + \mathrm{a}_{121211}\mathrm{u}_{113}\mathrm{u}_{123} + \mathrm{a}_{12126}\mathrm{u}_{123}\mathrm{u}_{62}.$$

Let $q=(u_{11},u_{21},u_{31},u_{41},u_{52},u_{62},u_{72},u_{82},u_{93},u_{103},u_{113},u_{123})$ and $p=(v_{11},v_{21},v_{31},v_{41},v_{52},v_{62},v_{72},v_{82},v_{93},v_{103},v_{113},v_{123})$ be the eigenvectors of N and N^T for the eigenvalue μ and its conjugate $\overline{\mu}$.

The coefficients of the normal form g_{20}, g_{11}, g_{02} are written by Neamțu and Opriș (2008):

$$g_{20}=B_{10}v_{103}+B_{11}v_{113}+B_{12}v_{123}+B_9v_{93};$$
$$g_{11}=C_{10}v_{103}+C_{11}v_{113}+C_{12}v_{123}+C_9v_{93};$$
$$g_{02}=D_{10}v_{103}+D_{11}v_{113}+D_{12}v_{123}+D_9v_{93};$$

In order to find the normal form of system (9), let μ bea complex conjugate solution with its modulus 1 of the characteristic equation.

Proposition 1.

(i) The solution of (9) in the neighbourhood the equilibtium point $(q_{10},q_{20},q_{30},q_{40})$is:

$$u_{93}[j]=q_{10}+u_{93}z[j-1]+\overline{u_{93}}\,\overline{z[j-1]}+\frac{w_{209}}{2z[j-1]^2}+w_{119}z[j-1]\overline{z[j-1]}+\frac{w_{029}}{\overline{2z[j-1]^2}};$$
$$u_{103}[j]=q_{20}+u_{103}z[j-1]+\overline{u_{103}}\,\overline{z[j-1]}+\frac{w_{2010}}{2z[j-1]^2}+w_{1110}z[j-1]\overline{z[j-1]}+\frac{w_{0210}}{\overline{2z[j-1]^2}};\ (30)$$
$$u_{113}[j]=q_{30}+u_{113}z[j-1]+\overline{u_{113}}\,\overline{z[j-1]}+\frac{w_{2011}}{2z[j-1]^2}+w_{1111}z[j-1]\overline{z[j-1]}+\frac{w_{0211}}{\overline{2z[j-1]^2}};$$
$$u_{123}[j]=q_{40}+u_{123}z[j-1]+\overline{u_{123}}\,\overline{z[j-1]}+\frac{w_{2012}}{2z[j-1]^2}+w_{1112}z[j-1]\overline{z[j-1]}+\frac{w_{0212}}{\overline{2z[j-1]^2}};$$

where $z[j]$ is a solution of:

$$z[j]=\mu z[j-1]+\frac{g_{20}}{2z[j-1]^2}+g_{11}z[j-1]\overline{z[j-1]}+\frac{g_{02}}{\overline{2z[j-1]^2}}+\frac{g_{21}}{2z[j-1]^2\,\overline{z[j-1]}};\ (31)$$

(ii) There is a change of complex variables so that the equation in $z[j]$ is:

$$w[j]=\mu w[j-1]+Cw[j-1]^2\,\overline{w[j-1]}+O\left|\overline{w[j-1]}\right|^4;\ (32)$$

called the normal form, where:

$$C=g_{20}g_{11}\frac{\overline{\mu}-3-2\mu}{2\mu^2-\overline{\mu}-1}+\frac{|g_{02}|}{2(\mu^2-\overline{\mu})}+\frac{g_{21}}{2};\ (33)$$

is the resonant cubic coefficient.

(iii) Consider $I_0 = \mathrm{Re}(\exp(-I\theta_0)C),\ \theta_0 = \arctan\left(\frac{\mathrm{Im}(\mu)}{\mathrm{Re}(\mu)}\right)$.

If $I_0 < 0$ in the neighborhood of the equilibrium point is a stable closed curve. *If* $I_0 > 0$ *the invariant closed curve is unstable.*

6. Numerical simulations

For the numerical simulations we consider the parameters:

$$a = 10,\ b = 0.6,\ c = 5,\ r = 0.4,\ \theta_1 = 0.65,\ \theta_2 = 0.2,\ S_1 = 1, S_2 = 0.1, S_3 = 300,$$
$$p = 30,\ \beta_1 = 0.0085, \beta_2 = 0.0015, \beta_3 = 0.003, \beta_4 = 0.0025.$$

UsingMaple2022 we obtain the coordinates of the equilibrium point:

$$q_{10} = 19.6205, q_{20} = 16.475, q_{30} = 36.5079, q_{40} = 14.4642.$$

The characteristic equation is given by:

$$eq_1 = z^{12} + 0.4669\ z^{11} + 0.0823\ z^{10} - 3.6526\ z^9 - 1.2541\ z^8 + 0.2223\ z^7 + 5.0092\ z^6 + \\ + 2.2095\ z^5 + 0.3915\ z^4 - 3.0469\ z^3 + 0.3195\ z^2 - 0.05693\, z + 0.6907.$$

The solutions of the characteristic equation are not all with the modulus less than 1, then the system is unstable.

For $\mu = 0.7570 + 0.6533\mathrm{i}$ we have $g_{21} == -0.000042 - 0.000108\mathrm{i}$ and $C == -0.00929 - 0.00689\mathrm{i}$. Then: $\theta_0 = -1.27075, I_0 = 0.003839 > 0$ and in the neighborhood of the equilibrium point there is a stable closed invariant curve.

For $\tau_3 = 3, \tau_2 = 2$, in Figure 1 we can visualize the oscillatory orbits for: the recycling prices of refurbished goods, collected by each agent to the consumer, and the recycling price of used products collected by agent 3 to the third-party:

$$(j, \mathrm{q}_1[j]),\ \ (j, \mathrm{q}_2[j]),\ \ (j, \mathrm{q}_3[j]), (j, \mathrm{q}_4[j])$$

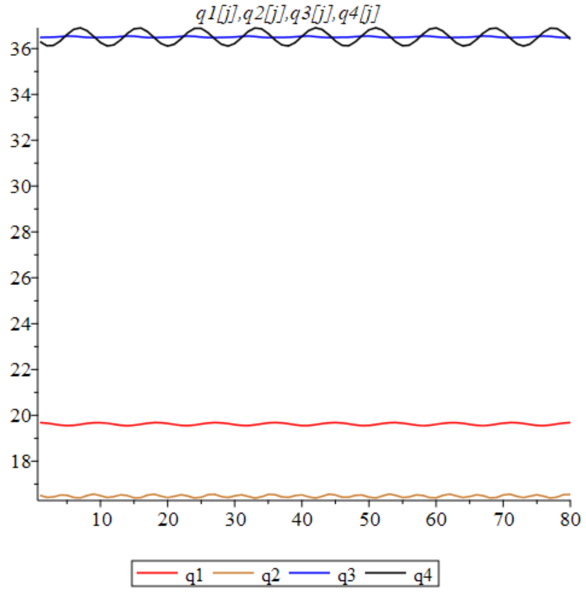

Figure 1. The orbits of system (9)$(j, q_1[j])$, $(j, q_2[j])$, $(j, q_3[j])$, $(j, q_4[j])$

Figure 2 represents the trajectories for the profits of: the online recycle, the offline recycle and agent the producer: $(j, P_1[j])$, $(j, P_2[j])$, $(j, P_3[j])$).

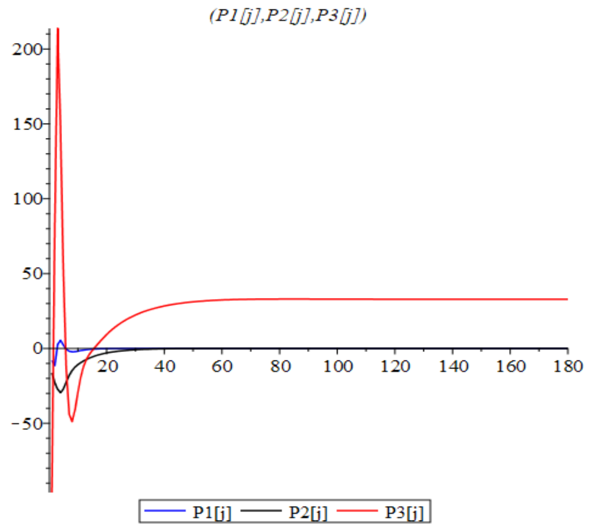

Figure 2. The representation of the profits of the economic agents $(j, P_1[j])$, $(j, P_2[j])$, $(j, P_3[j])$, $(j, P_4[j])$

7. Conclusions

In the present paper, based on the existing literature, a recycling model was built described by a nonlinear system of equations with discrete time and time delays. The closed-loop supply chain included customers and three types of economic agents such as: the online recycle (agent 1), the offline recycle (agent 2), and the producer (agent 3). Three ways for customers to return used goods were employed: directly to the manufacturer, an online or offline recycler, or a third party.

In order to design our mathematical model, we considered that the economic agents make decisions based on a prior time period, within the framework of a repeated game, where the marginal profit was taken into account. If the marginal profit in time period j-1 is positive, the approach will be continued in time period j. However, the process needs to be adjusted in period j due to the negative marginal profit. For the discrete mathematical model, four variables were considered: the recycling prices of refurbished goods, collected by each agent to the consumer, and the recycling price of used products collected by agent 3 to the third-party.

A complex dynamic analysis was conducted with the aim of studying the sustainability of the system. Thus, fixed values were established for the two delays considered in the mathematical model. The Möbius transformation was used to determine the roots in the modulus equal to 1 for the characteristic polynomial of degree 12. These results can be used for the study of discrete dynamic equations with reinforcements. It was revealed that the characteristic equation, corresponding to the associated linear system, has two pairs of complex conjugate roots with their modulus equal to 1. For a pair of complex conjugate roots, the dynamics of the system is simulated, and it is found that it is sustainable. Numerical simulations for the study of sustainability are considered to illustrate the theoretical results.

The issue of sustainability remains topical with challenges for the economics, biology, and the environment. As future research, we propose to identify the real economic parameters that improve the economic interpretations of our mathematical model. Also, we will consider the stochastic aspect taking into account the approach from (Mircea et al., 2011).

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